Maximum marks: 100   Time: 3 hours

1. Show that the ordered field $\mathbb{R}$ of real numbers has the Archimedean property.   [15]

2. For any two subsets A, B of $\mathbb{R}$, define

$$A + B = \{a + b : a \in A, b \in B\};$$

$$AB = \{ab : a \in A, b \in B\}.$$

   Show that if A, B are bounded above,

$$\sup(A + B) = \sup(A) + \sup(B).$$

   Give an example to show that

$$\sup(AB) = \sup(A).\sup(B),$$

   need not be true.   [15]

3. Show that intervals [0,1] and (0,1) have same cardinality.   [15]

4. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers converging to $a \in \mathbb{R}$. Define

$$b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

   for $n \geq 1$. Show that $\{b_n\}_{n \geq 1}$ is a convergent sequence converging to a.   [15]

5. Show that every Cauchy sequence of real numbers is convergent.   [15]

6. Let A be a subset of $\mathbb{R}$. Show that interior of A is the largest open set contained in A.   [15]

7. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define a new function $g_+ : \mathbb{R} \to \mathbb{R}$ by

$$g_+(x) = \begin{cases} 0 & \text{if } g(x) < 0, \\ g(x) & \text{otherwise.} \end{cases}$$

   Show that if g is continuous then g_+ is continuous. Give an example where g_+ is continuous but g is not continuous.   [15]